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A FAMILY OF APPROXIMATE SOLUTIONS AND EXPLICIT ERROR ESTIMATES FOR THE NONLINEAR STATIONARY NAVIER-STOKES PROBLEM

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A FAMILY OF APPROXIMATE SOLUTIONS AND EXPLICIT ERROR ESTIMATES FOR THE NONLINEAR STATIONARY NAVIER-STOKES PROBLEM

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Abstract. An algorithm for solving the nonlinear stationary Navier-Stokes problem is developed. Explicit error estimates are given.

1. Introduction.

Since the separation problem of aerodynamics is at present intractable, it has been decided to undertake a closely related problem via a mathematical technique that is potentially adaptable to the separation problem. Specifically, the problem under consideration is the "nonlinear stationary Navier-Stokes problem" of fluid dynamics. The generalized Newton's method, as developed by Kantorovich [5, 7] is used. Its application to this problem is of definite value for those seeking practical solutions of related fluid flow problems. The following questions are considered:

(i) Under what conditions does the sequence of functions obtained by Newton's method converge to the solution?

(ii) How should the initial guess be made, as a function of ν, so as to guarantee convergence?

(iii) At what rate does the sequence of approximate solutions converge?

Given S, a two-dimensional Green's domain, and $f_1(x,y) \in C^1(S)$, $f_2(x,y) \in C^1(S)$, the nonlinear stationary Navier-Stokes problem is:

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$$
(1) \qquad \left\{
\begin{array}{l}
uu_x + vu_y + q_x - \nu\,\Delta u + f_1(x,y) = 0\ , \\
uv_x + vv_y + q_y - \nu\,\Delta v + f_2(x,y) = 0\ , \\
\qquad\qquad\qquad\qquad u_x + v_y = 0\ , \\
\text{with boundary conditions} \\
\quad u(\partial S) = -b_2(\partial S)\ ,\quad v(\partial S) = b_1(\partial S)\ .
\end{array}
\right.
$$

An equivalent expression is

$$
(2) \qquad P(\psi) \equiv \nu\Delta\Delta\psi + \psi_y\Delta\psi_x - \psi_x\Delta\psi_y + f_{1_y} - f_{2_x} = 0
$$

$$
\psi_x(\partial S) = b_1(\partial S)\ ,\quad \psi_y(\partial S) = b_2(\partial S)\ ,
$$

where

$$
\psi_x = v\ ,\quad \psi_y = -u\ ,
$$

and P is a mapping from $C^4(S)$ into $C^0(S)$ with norm

$$
\|\psi\|_{C^N} = \sum_{n=0}^{N}\sum_{m=0}^{n} \max\left|\frac{\partial^n\psi}{\partial x^m\partial y^{n-m}}\right|\ .
$$

For clarity, the equivalence of (1) and (2) is now shown.

LEMMA (1) $\leftrightarrow$ (2).

Proof.

(1) $\rightarrow$ (2) directly follows from [6, Theorem 6, p. 131].

(2) $\rightarrow$ (1): (2) can be readily rearranged into the form

$$
(-\nu\Delta u + uu_x + vu_y + f_1)_y = (-\nu\Delta v + uv_x + vv_y + f_2)_x\ ,
$$

with

$$
u = \psi_y\ ,\quad v = -\psi_x\ .
$$

Let

$$
\vec{Z} = (-\nu\Delta u + uu_x + vu_y + f_1)\vec{i} + (-\nu\Delta v + uv_x + vv_y + f_2)\vec{j}\ .
$$

Note the fact: if $\vec{V} = a\vec{i} + b\vec{j} \in C^1(S)$, then

$$
\exists\, F \ni \vec{V} = \nabla F \leftrightarrow a_y = b_x\ .
$$

Hence, $\exists\ q \ni \vec{Z} = \nabla q$.

Therefore, $(2) \rightarrow (1)$.

2. Main Results. We seek a solution to (2) by the generalized Newton's method. Consider the equation

$$P(\psi_0) + P'(\psi_0)(\psi - \psi_0) = 0\ .$$

If ψ_1 is a solution of this equation, we can write a new equation

$$P(\psi_1) + P'(\psi_1)(\psi - \psi_1) = 0\ .$$

Assume that for each $n \geq 0\ \exists\ \psi_{n+1} \ni$

$$\begin{aligned} &P(\psi_n) + P'(\psi_n)(\psi_{n+1} - \psi_n) = 0\ , \\ &\psi_{n_x}(\partial S) = b_1\ , \quad \psi_{n_y}(\partial S) = b_2\ . \end{aligned} \tag{3}$$

(See LEMMA 1 for explicit expression for P'.)

If $\lim\limits_{n\to\infty} \psi_n$ exists, let ψ^* be the limit. Then

$$P(\psi^*) + P'(\psi^*)(\psi^* - \psi^*) = 0$$

$$P(\psi^*) = 0\ .$$

Thus ψ^* is the desired solution. This is Newton's method.

Define

$$H_N = \max_{x',y' \in S} \int \sum_{n=0}^{N} \sum_{m=0}^{n} \left| \frac{\partial^n G(x',y',x,y)}{\partial x'^n \partial y'^{n-m}} \right| dx\, dy$$

where G is the Green's function of the problem

$$\Delta\Delta\tilde{\psi} = 0 \text{ in } S\ , \quad \tilde{\psi}(\partial S) = 0\ , \quad \tilde{\psi}_n(\partial S) = 0\ .$$

[1], [2], [3], [4], [8].

Let $M_{\psi_0} = \max(\max|\Delta\psi_{0_y}|,\ \max|\Delta\psi_{0_x}|,\ \max|\psi_{0_x}|,\ \max|\psi_{0_y}|)$.

Therefore, based on the remarkable theory developed by Kantorovich, we obtain the following result:

THEOREM 1. *If the initial* ψ_0 *is* $\nrightarrow$

$$\psi_{0_x}\Big|_{\partial S} = b_1, \quad \psi_{0_y}\Big|_{\partial S} = b_2, \quad M_{\psi_0}H_3 < \nu ,$$

and

$$\|P(\psi_0)\|_{C^0} \le \frac{\nu^2\left(1 - \frac{1}{\nu} M_{\psi_0}H_3\right)^2}{2H_4^2} ,$$

then the Newton-Kantorovich sequence $\{\psi_m\}_{m=0}^{\infty}$ *does in fact converge to the unique solution of (2).*

Proof. By the theorem of Kantorovich [5, p. 708], it is sufficient to show that

$$\|P(\psi_0)\| \le \frac{1}{2\|P'(\psi_0)^{-1}\|^2\|P''\|} ;$$

this is shown by the following lemmas.

LEMMA 1. $P'(\psi)$, *the Frechet derivative exists at all points* ψ_0 *in the domain, and*

$$P'(\psi_0)g = \nu\Delta\Delta g + \psi_{0_y}\Delta g_x + \Delta\psi_{0_x}g_y - \Delta\psi_{0_y}g_x - \psi_{0_x}\Delta g_y .$$

Proof:

$$P(\psi) = \nu\Delta\Delta\psi + \psi_y\Delta\psi_x - \psi_x\Delta\psi_y + f_{1_y} - f_{2_x} .$$

If

$$\lim_{\|a\|\to 0} \frac{\|P(\psi+a) - P(\psi) - La\|}{\|a\|} = 0 ,$$

for some linear operator L, then define $P'(\psi_0) = L$.

$$P(\psi_0+a) - P(\psi_0) = \nu\Delta\Delta a + \psi_{0_y}\Delta a_x + a_y\Delta\psi_{0_x} + a_y\Delta a_x - a_x\Delta\psi_{0_y} - \psi_x\Delta a_y - a_x\Delta a_y .$$

Let

$$L(\psi_0)(a) = \nu\Delta\Delta a + \psi_{0_y}\Delta a_x + a_y\Delta\psi_{0_x} - a_x\Delta\psi_{0_y} - \psi_{0_x}\Delta a_y .$$

Then $L(\psi_0)$ is a linear operator. Therefore, $L(\psi) = P'(\psi)$ if

$$\lim_{\|a\|\to 0} \frac{\|P(\psi_0+a) - P(\psi_0) - La\|}{\|a\|} = \lim_{\|a\|\to 0} \frac{\|\varepsilon_y\Delta a_x - a_x\Delta a_y\|}{\|a\|} = 0 .$$

$$a_y\Delta a_x - a_x\Delta a_y \ \varepsilon\ C^0 .$$

Therefore,

$$\|a_y \Delta a_x - a_x \Delta a_y\| = \max_S |a_y \Delta a_x - a_x \Delta a_y| .$$

$$a \in C^4 .$$

Therefore,

$$\|a\| = \sum_{n=0}^{4} \sum_{m=0}^{n} \max \left| \frac{\partial^n a}{\partial x^m \partial y^{n-m}} \right| .$$

$$|a_y \Delta a_x - a_x \Delta a_y| \leq |a_y|(|a_{xxx}| + |a_{xyy}|) + |a_x|(|a_{xxy}| + |a_{yyy}|) .$$

Therefore,

$$\max_S |a_y \Delta a_x - a_x \Delta a_y| \leq \max_S |a_y| \left(\max_S |a_{xxx}| + \max_S |a_{xyy}| \right) + \max_S |a_x| \left(\max_S |a_{xxy}| + \max_S |a_{yyy}| \right) .$$

So:

$$0 \leq \lim_{\|a\| \to 0} \frac{\|P(\psi_0 + a) - P(\psi_0) - La\|}{\|a\|}$$

$$\leq \lim_{\|a\| \to 0} \frac{\max_S |a_y| \left(\max_S |a_{xxx}| + \max_S |a_{xyy}| \right) + \max_S |a_x| \left(\max_S |a_{xxy}| + \max_S |a_{yyy}| \right)}{a}$$

$$\leq \lim_{\|a\| \to 0} \frac{\|a\|(\|a\| + \|a\|) + \|a\|(\|a\| + \|a\|)}{\|a\|} .$$

Therefore,

$$\lim_{\|a\| \to 0} \frac{\|P(\psi_0 + a) - P(\psi_0) - La\|}{\|a\|} = 0 . \qquad \text{Q.E.D.}$$

LEMMA 2. $P''(\psi)$ *exists at all points* ψ_0 *in* $C^4(S)$ *and*

$$P''(\psi_0)\psi\phi = \psi_y \Delta\phi_x + \phi_y \Delta\psi_x - \phi_x \Delta\psi_y - \psi_x \Delta\phi_y .$$

Proof. By definition, $P''(\psi_0)$ exists if there is a bilinear operator $B \ni$

$$\lim_{\|\phi\| \to 0} \frac{\|P'(\psi_0 + \phi) - P'(\psi_0) - B\phi\|}{\|\phi\|} = 0 .$$

If so, $P''(\psi_0)$ is defined to be B.

$$P'(\psi_0+\phi)g = \nu\Delta\Delta g + (\psi_0+\phi)_y\Delta g_x + g_y\Delta(\psi_0+\phi)_x - g_x\Delta(\psi_0+\phi)_y - (\psi_0+\phi)_x\Delta g_y .$$

$$[P'(\psi_0+\phi) - P'(\psi_0)]g = \phi_y\Delta g_x + g_y\Delta\phi_x - g_x\Delta\phi_y - \phi_x\Delta g_y .$$

Now,

$$[P'(\psi_0+\phi) - P'(\psi_0)](g_1+g_2) = [P'(\psi_0+\phi) - P'(\psi_0)]g_1 + [P'(\psi_0+\phi) - P'(\psi_0)]g_2 ,$$

and

$$[P'(\psi_0+\phi_1+\phi_2) - P'(\psi_0)]g = [P'(\psi_0+\phi_1) - P'(\psi_0)]g + [P'(\psi_0+\phi_2) - P'(\psi_0)]g .$$

Therefore, $[P'(\psi_0+\phi) - P'(\psi_0)]g$ is a bilinear operator of ϕ and g. Let $B\phi = P'(\psi_0+\phi) - P'(\psi_0)$. Then it follows that

$$P''(\psi_0)\phi\theta = \phi_y\Delta\theta_x + \theta_y\Delta\phi_x - \theta_x\Delta\phi_y - \phi_x\Delta\theta_y .$$

LEMMA 3. $P'(\psi_0)^{-1}$ *exists.*

Proof. Given $P(\psi_0) + P'(\psi_0)(\psi-\psi_o) = 0$, let $\tilde{\psi} = \psi - \psi_o$, then

$$P'(\psi_0)\tilde{\psi} = -P(\psi_0) .$$

Equivalently, $\nu\Delta\Delta\tilde{\psi} + \Delta\psi_{0x}\tilde{\psi}_y + \psi_{0y}\Delta\tilde{\psi}_x - \psi_{0x}\Delta\tilde{\psi}_y - \Delta\psi_{0y}\tilde{\psi}_x = -P(\psi_0)$. This equation can be abbreviated as $\nu\Delta\Delta\tilde{\psi} = \tilde{f}(\tilde{\psi}) + F$. Let G be the Green's function for $\tilde{\psi}$ (see Theorem 1), then

$$\tilde{\psi} = \frac{1}{\nu}\int G\tilde{f} + \frac{1}{\nu}\int GF .$$

Define the linear operators

$$A\tilde{\psi} = \frac{1}{\nu}\int G\tilde{f}(\tilde{\psi}) .$$

$$B[-P(\psi_0)] = \frac{1}{\nu}\int GF .$$

Then

$$\left(I - \frac{1}{\nu}A\right)\tilde{\psi} = B[-P(\psi_0)] .$$

Under proper conditions, as shown later in the proof, $\left(I - \frac{1}{\nu}A\right)^{-1}$ exists. Then

$$\tilde{\psi} = \left(I - \frac{1}{\nu} A\right)^{-1} B[-P(\psi_0)] \ .$$

Therefore,

$$P'(\psi_0)\left(I - \frac{1}{\nu} A\right)^{-1} B[-P(\psi_0)] = -P(\psi_0) \ ;$$

$$P'(\psi_0)\left(I - \frac{1}{\nu} A\right)^{-1} B = I \ ,$$

and $\left(I - \frac{1}{\nu} A\right)^{-1} B$ is a right inverse. Also, $\left(I - \frac{1}{\nu} A\right)^{-1} BP'(\psi_0)\tilde{\psi} = \tilde{\psi}$; $\left(I - \frac{1}{\nu} A\right)^{-1} BP'(\psi_0) = I$, and $\left(I - \frac{1}{\nu} A\right)^{-1} B$ is a left inverse. Therefore,

$$\left(I - \frac{1}{\nu} A\right)^{-1} B = [P'(\psi_0)]^{-1} \ .$$

We now show the conditions under which $\left(I - \frac{1}{\nu} A\right)^{-1}$ exists.

$$A\psi = \int_S G(\psi_x \Delta\psi_{0_y} - \psi_y \Delta\psi_{0_x} + \psi_{0_x} \Delta\psi_y - \psi_{0_y} \Delta\psi_x) dS$$

$$\|A\| = \sup_{\|\psi\| \le 1} \|A\psi\|$$

so

$$\|A\|_C = \sup_{\|\psi\| \le 1} \left[\max_{x',y' \epsilon S} \left| \int_S G(\psi_x\Delta\psi_{0_y} - \psi_y\Delta\psi_{0_x} + \psi_{0_x}\Delta\psi_y - \psi_{0_y}\Delta\psi_x) dS \right| + \max (\ )_{x'} \ + \ . \ . \ . \right.$$

$$\|A\|_C \le \sup_{\|\psi\| \le 1} \left| \max_{x',y' \epsilon S} \int [\,|G| + |G_{x'}| + |G_{y'}| + \ . \ . \ .] \cdot [\,|\psi_x||\Delta\psi_{0_y}| + \ . \ . \ .] \right.$$

$$\|A\|_C \le \max_{\|\psi\| \le 1} \left(\max_{x',y' \epsilon S} M_{\psi_0} \int [\,|G| + |G_{x'}| + |G_{y'}| + \ . \ . \ .] \cdot [\,|\psi_x| + |\psi_y| + \ . \ . \ .] \right) .$$

Therefore,

$$\|A\|_{C^4} \le M_{\psi_0}\left(\max \int [\,|G| + |G_{x'}| + |G_{y'}| + \ . \ . \ .] dx\, dy \right) .$$

$$\|A\|_{C^4} \le M_{\psi_0} H_3 \ .$$

Therefore, $\left(I - \frac{1}{\nu} A\right)^{-1}$ exists if $M_{\psi_0} H_3 < \nu$. $\left(I - \frac{1}{\nu} A\right)^{-1} = \sum_{n=0}^{\infty} \frac{1}{\nu^n} A^n$ exists.

LEMMA 4. $\|P''(\psi_0)\| \le 1$.

Proof.

$$\|P''(\psi_0)\| = \max_{\substack{\|\phi\| \le 1 \\ \|\theta\| \le 1}} |\phi_y \Delta\theta_x + \theta_y \Delta\phi_x - \phi_x \Delta\theta_y - \theta_x \Delta\phi_y|$$

$$\le \max(|\phi_y| + |\phi_{xxx}| + |\phi_{xyy}| + |\phi_x| + |\phi_{xxy}| + |\phi_{yyy}|) \le 1 .$$

This completes the proof of Theorem 1.

THEOREM 2. *Under the hypothesis of Theorem 1, the error estimate for the mth approximate solution is expressed by*

$$\|\psi - \psi_m\| \le \left(2^{2^m - m}\right) \left[\frac{H_4}{\nu\left(1 - \frac{1}{\nu} M_{\psi_0} H_3\right)}\right]^{2^{m+1}-1} \|P(\psi_0)\|^{2m} .$$

Proof. By the Kantorovich theory, this result follows from the hypothesis of Theorem 1.

COROLLARY 1. *As a function of* ν, *for fixed* m,

$$\|\psi - \psi_m\| = 0\left(\nu^{-2^{m+1}+1}\right) .$$

COROLLARY 2. *If the hypothesis of Theorem 1 is satisfied, then for a given* $\epsilon > 0$, *there exists a denumerably infinite number of linear equations and solutions* ψ_m *as specified by (3) such that the entire family of* ψ_m*'s are within the* ϵ*-neighborhood of the exact solution* ψ *of (2), i.e.,*

$$\|\psi - \psi_n\|_{C^4} < \epsilon .$$

Proof. Follows directly from Theorem 2.

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